SOME PROPERTIES OF HYPERBOLIC GENERALIZED TRIBONACCI QUATERNIONS

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Abstract

In this paper, we introduced the hyperbolic generalized tribonacci quaternions, $HW_n = W_n + W_{n+1}\boldsymbol{j_1} + W_{n+2}\boldsymbol{j_2} + W_{n+3}\boldsymbol{j_3}$, $n \geq 0$, W_n is the generalized tribonacci number. Several properties of these hyperbolic quaternions are investigated, including the Binet formulas, generating functions, and finite summation formula. Our results extend and generalize well-known theorems.

1. Introduction

The Fibonacci sequence is a series of numbers in which each number is the sum of the two that precede it. Let F_n denote the Fibonacci sequence, which is defined by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1.1}$$

where $F_0 = 0$ and $F_1 = 1$.
That is, the Fibonacci sequence is $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$.

Horadam [9] introduced the (p,q)-Fibonacci sequence as a generalized form of the Fibonacci numbers, defined for positive integer values of p and q. the (p,q)-Fibonacci sequence is defined by the following recurrence relation:

$$\mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}, \quad n \geq 2 \tag{1.2}$$

where $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$.
If we set $p = 1 = q$, then $\{\mathcal{F}_n\}$ is the well-known Fibonacci sequence.
If we set $p = k$ and $q = 1$, then $\{\mathcal{F}_n\}$ is the well-known k-Fibonacci sequence.
That is, the (p,q)-Fibonacci sequence is $\{0, 1, p, p^2+q, p^3+2pq, p^4+3p^2q+q^2, p^5+4p^3q+3pq^2, p^6+5p^4q+6p^2q^2+q^3, p^7+6p^5q+10p^3q^2+4pq^3, \ldots\}$.

Subsequently, the tribonacci sequence, a generalized form of the Fibonacci sequence, was introduced. It consists of numbers where each is the sum of its three preceding numbers. Tribonacci sequence is defined by the following recurrence relation:

$$\mathfrak{F}_n = \mathfrak{F}_{n-1} + \mathfrak{F}_{n-2} + \mathfrak{F}_{n-3}, \quad n \geq 3 \tag{1.3}$$

where $\mathfrak{F}_0 = \mathfrak{F}_1 = 0$ and $\mathfrak{F}_2 = 1$,
That is, the tribonacci sequence is $\{0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots\}$.

The Narayana numbers, named after the 14th-century Indian mathematician Tadepalli Venkata Narayana, Narayana sequence is defined by the following recurrence relation:

$$N_n = N_{n-1} + N_{n-3}, \quad n \geq 3 \tag{1.4}$$

where $N_0 = 0$, $N_1 = 1$, and $N_2 = 1$.
That is, the Narayana sequence is $\{0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \ldots\}$.

In 1902, Macfarlane [12] introduced hyperbolic quaternions, which, unlike real quaternions, lack commutativity, and conducted a study on their properties. Hyperbolic quaternions are utilized across diverse fields such as physics, computer graphics, and geometric algebra [4, 15]. A set of hyperbolic quaternions is represented as

$$\mathbf{H} = \{q = q_0 + q_1\boldsymbol{j_1} + q_2\boldsymbol{j_2} + q_3\boldsymbol{j_3} | \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where $\boldsymbol{j_1}, \boldsymbol{j_2}$ and $\boldsymbol{j_3}$ are hyperbolic quaternion units satisfying the conditions

$$\boldsymbol{j_1^2} = \boldsymbol{j_2^2} = \boldsymbol{j_3^2} = \boldsymbol{j_1 j_2 j_3} = 1,\ \boldsymbol{j_1 j_2} = \boldsymbol{j_3} = -\boldsymbol{j_2 j_1},\ \boldsymbol{j_2 j_3} = \boldsymbol{j_1} = -\boldsymbol{j_3 j_2},\ \boldsymbol{j_3 j_1} = \boldsymbol{j_2} = -\boldsymbol{j_1 j_3}.$$

Let $\mathbf{h}_0 = a_0 + b_0\boldsymbol{j_1} + c_0\boldsymbol{j_2} + d_0\boldsymbol{j_3}$ and $\mathbf{h}_1 = a_1 + b_1\boldsymbol{j_1} + c_1\boldsymbol{j_2} + d_1\boldsymbol{j_3}$ denote two hyperbolic quaternions. Equality, addition, subtraction, and scalar multiplication and multiplication can be defined as follows:

$$\begin{aligned}
\mathbf{h}_0 = \mathbf{h}_1 \ &\Leftrightarrow \ a_0 = a_1, b_0 = b_1, c_0 = c_1, d_0 = d_1;\\
\mathbf{h}_0 + \mathbf{h}_1 &= (a_0 + a_1) + (b_0 + b_1)\boldsymbol{j_1} + (c_0 + c_1)\boldsymbol{j_2} + (d_0 + d_1)\boldsymbol{j_3};\\
\mathbf{h}_0 - \mathbf{h}_1 &= (a_0 - a_1) + (b_0 - b_1)\boldsymbol{j_1} + (c_0 - c_1)\boldsymbol{j_2} + (d_0 - d_1)\boldsymbol{j_3};\\
\lambda\mathbf{h}_0 &= \lambda a_0 + \lambda b_0\boldsymbol{j_1} + \lambda c_0\boldsymbol{j_2} + \lambda d_0\boldsymbol{j_3}, \quad \lambda \in \mathbb{R};\\
\mathbf{h}_0\mathbf{h}_1 &= (a_0a_1 + b_0b_1 + c_0c_1 + d_0d_1) + (a_0b_1 + b_0a_1 + c_0d_1 - d_0c_1)\boldsymbol{j_1}\\
&\quad + (a_0c_1 - b_0d_1 + c_0a_1 + d_0b_1)\boldsymbol{j_2} + (a_0d_1 + b_0c_1 - c_0b_1 + d_0a_1)\boldsymbol{j_3}.
\end{aligned}$$

Then, the set $\mathbf{H}$ is a vector space over $\mathbb{R}$. Moreover, the conjugate of a hyperbolic quaternion is established by

$$\bar{h} = a - b\boldsymbol{j_1} - c\boldsymbol{j_2} - d\boldsymbol{j_3}.$$

In this paper, we introduced the hyperbolic generalized tribonacci quaternions. Several properties of these quaternions are investigated, including the Binet formulas, generating functions, and summation formula. Our results extend and generalize well-known theorems [1, 16].

This study aims to introduce the hyperbolic generalized tribonacci quaternions. We demonstrate that this new hyperbolic quaternion sequence encompasses previously established sequences such as hyperbolic tribonacci quaternions, hyperbolic (p, q)-Fibonacci quaternions, and hyperbolic Narayana quaternions. Additionally, we offer the generating function and Binet's formula for hyperbolic generalized tribonacci quaternions, along with a summation formula.

2. Preliminaries

In 2017, Cerda-Morales [2] defined and provided the binet formula, summation formula of generalized tribonacci numbers as follows:

Definition 2.1. [2] The generalized tribonacci sequence, $\{W_n\}$ defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3},\ n \geq 3, \tag{2.1}$$

where $W_0 = a$, $W_1 = b$, $W_2 = c$ are integers and r, s, t, are real numbers.

Many authors have examined this sequence (see, for example, [3, 5, 13, 18]). As the elements of this tribonacci-type number sequence provide third order iterative relation, its characteristic equation is $x^3 - rx^2 - sx - t = 0$, whose roots are [2] $\alpha = \alpha(r,s,t) = \frac{r}{3} + A + B$, $\beta = \frac{r}{3} + \omega A + \omega^2 B$ and $\gamma = \frac{r}{3} + \omega^2 A + \omega B$, where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{\frac{1}{3}},\ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{\frac{1}{3}},$$

with $\Delta = \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}$ and $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$.

In fact, the generalized tribonacci sequence is the generalization of the well-known sequences like Fibonacci, k-Fibonacci, (p,q)-Fibonacci, tribonacci and Narayana sequences.

Table 1. represents several numbers of this family according to initial values and r, s, t values

Name	$\{W_n\} = \{W_n(a,b,c,r,s,t)\}$	Recurrence Relation
Fibonacci	$\{F_n\} = \{W_n(0,1,1,1,1,0)\}$	$F_n = F_{n-1} + F_{n-2}$
k-Fibonacci	$\{F_{k,n}\} = \{W_n(0,1,k,k,1,0)\}$	$F_{k,n} = kF_{k,n-1} + F_{k,n-2}$
(p,q)-Fibonacci	$\{\mathcal{F}_n\} = \{W_n(0,1,p,p,q,0)\}$	$\mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}$
tribonacci	$\{\mathfrak{F}_n\} = \{W_n(0,0,1,1,1,1)\}$	$\mathfrak{F}_n = \mathfrak{F}_{n-1} + \mathfrak{F}_{n-2} + \mathfrak{F}_{n-3}$
Narayana	$\{N_n\} = \{W_n(0,1,1,1,0,1)\}$	$N_n = N_{n-1} + N_{n-3}$

Theorem 2.2. [2] *The Binet formula for the generalized tribonacci numbers can be expressed as:*

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} - \frac{d_2\beta^n}{(\alpha-\beta)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\alpha-\gamma)(\beta-\gamma)}, \tag{2.2}$$

where $d_1 = W_2 - (\beta+\gamma)W_1 + (\beta\gamma)W_0$, $d_2 = W_2 - (\alpha+\gamma)b + (\alpha\gamma)W_0$ and $d_3 = W_2 - (\alpha+\beta)b + (\alpha\beta)W_0$.

Theorem 2.3. [2] *The summation of the first n generalized tribonacci numbers:*

$$\sum_{i=0}^{n} W_i = \frac{1}{\delta}(W_{n+2} + (1-r)W_{n+1} + tW_n + (r+s-1)a + (r-1)b - c), \tag{2.3}$$

where $\delta = \delta(r,s,t) = r + s + t - 1$.

3. Main Results

In this part, we will initially provide the definition of the hyperbolic generalized tribonacci quaternions, we will explore various properties associated with these hyperbolic quaternions.

Definition 3.1. Let $n \geq 0$ be an integer. The n-th hyperbolic generalized tribonacci quaternions is defined by

$$HW_n = W_n + W_{n+1}\boldsymbol{j_1} + W_{n+2}\boldsymbol{j_2} + W_{n+3}\boldsymbol{j_3} \tag{3.1}$$

where W_n is the tribonacci number and $\boldsymbol{j_1}, \boldsymbol{j_2}, \boldsymbol{j_3}$ satisfy equalities
$\boldsymbol{j_1^2} = \boldsymbol{j_2^2} = \boldsymbol{j_3^2} = \boldsymbol{j_1 j_2 j_3} = 1,\ \boldsymbol{j_1 j_2} = \boldsymbol{j_3} = -\boldsymbol{j_2 j_1},\ \boldsymbol{j_2 j_3} = \boldsymbol{j_1} = -\boldsymbol{j_3 j_2},\ \boldsymbol{j_3 j_1} = \boldsymbol{j_2} = -\boldsymbol{j_1 j_3}.$

The first few terms of hyperbolic generalized tribonacci quaternions

$$\begin{aligned}
HW_0 &= W_0 + W_1\boldsymbol{j_1} + W_2\boldsymbol{j_2} + W_3\boldsymbol{j_3},\\
&= a + b\boldsymbol{j_1} + c\boldsymbol{j_2} + (rc + sb + ta)\boldsymbol{j_3},\\
HW_1 &= W_1 + W_2\boldsymbol{j_1} + W_3\boldsymbol{j_2} + W_4\boldsymbol{j_3},\\
&= b + c\boldsymbol{j_1} + (rc + sb + ta)\boldsymbol{j_2} + [r(rc + sb + ta) + sc + tb)]\boldsymbol{j_3},\\
HW_2 &= W_2 + W_3\boldsymbol{j_1} + W_4\boldsymbol{j_2} + W_5\boldsymbol{j_3},\\
&= c + (rc + sb + ta)\boldsymbol{j_1} + [r(rc + sb + ta) + sc + tb]\boldsymbol{j_2}\\
&\quad + [r^2(rc + sb + ta) + r(sc + tb) + s(rc + sb + ta) + tc]\boldsymbol{j_3},\\
HW_3 &= W_3 + W_4\boldsymbol{j_1} + W_5\boldsymbol{j_2} + W_6\boldsymbol{j_3},\\
&= (rc + sb + ta) + [r(rc + sb + ta) + sc + tb]\boldsymbol{j_1}\\
&\quad + [r^2(rc + sb + ta) + r(sc + tb) + s(rc + sb + ta) + tc]\boldsymbol{j_2}\\
&\quad + [r^3(rc + sb + ta) + r^2(sc + tb) + rs(rc + sb + ta) + rtc\\
&\quad + sr(rc + sb + ta) + s^2c + stb + t(rc + sb + ta)]\boldsymbol{j_3},\\
HW_4 &= W_4 + W_5\boldsymbol{j_1} + W_6\boldsymbol{j_2} + W_7\boldsymbol{j_3},\\
&= [r(rc + sb + ta) + sc + tb]\\
&\quad + [r^2(rc + sb + ta) + r(sc + tb) + s(rc + sb + ta) + tc]\boldsymbol{j_1}\\
&\quad + [r^3(rc + sb + ta) + r^2(sc + tb) + rs(rc + sb + ta) + rtc\\
&\quad + sr(rc + sb + ta) + s^2c + stb + t(rc + sb + ta)]\boldsymbol{j_2}\\
&\quad + [r^4(rc + sb + ta) + r^3(sc + tb) + r^2s(rc + sb + ta) + r^2tc\\
&\quad + sr^2(rc + sb + ta) + r(s^2c + stb) + rt(rc + sb + ta)\\
&\quad + sr^2(rc + sb + ta) + sr(sc + tb) + s^2(rc + sb + ta) + stc\\
&\quad + tr(rc + sb + ta) + tsc + t^2b]\boldsymbol{j_3},\\
&\ \ \vdots
\end{aligned}$$

Next, we present the recurrence relations of hyperbolic generalized tribonacci quaternions.

Lemma 3.2. *Assume $n \geq 3$, Then*

$$HW_n = rHW_{n-1} + sHW_{n-2} + tHW_{n-3}, \tag{3.2}$$

Proof. Using (2.1) and (3.1), we obtain

$$\begin{aligned}
HW_n &= W_n + W_{n+1}\boldsymbol{j_1} + W_{n+2}\boldsymbol{j_2} + W_{n+3}\boldsymbol{j_3} \\
&= (rW_{n-1} + sW_{n-2} + tW_{n-3}) + (rW_n + sW_{n-1} + tW_{n-2})\boldsymbol{j_1} \\
&\quad + (rW_{n+1} + sW_n + tW_{n-1})\boldsymbol{j_2} + (rW_{n+2} + sW_{n+1} + tW_n)\boldsymbol{j_3} \\
&= rW_{n-1} + sW_{n-2} + tW_{n-3} + rW_n\boldsymbol{j_1} + sW_{n-1}\boldsymbol{j_1} + tW_{n-2}\boldsymbol{j_1} \\
&\quad + rW_{n+1}\boldsymbol{j_2} + sW_n\boldsymbol{j_2} + tW_{n-1}\boldsymbol{j_2} + rW_{n+2}\boldsymbol{j_3} + sW_{n+1}\boldsymbol{j_3} + tW_n\boldsymbol{j_3} \\
&= (rW_{n-1} + rW_n\boldsymbol{j_1} + rW_{n+1}\boldsymbol{j_2} + rW_{n+2}\boldsymbol{j_3}) \\
&\quad + (sW_{n-2} + sW_{n-1}\boldsymbol{j_1} + sW_n\boldsymbol{j_2} + sW_{n+1}\boldsymbol{j_3}) \\
&\quad + (tW_{n-3} + tW_{n-2}\boldsymbol{j_1} + tW_{n-1}\boldsymbol{j_2} + tW_n\boldsymbol{j_3}) \\
&= r(W_{n-1} + W_n\boldsymbol{j_1} + W_{n+1}\boldsymbol{j_2} + W_{n+2}\boldsymbol{j_3}) \\
&\quad + s(W_{n-2} + W_{n-1}\boldsymbol{j_1} + W_n\boldsymbol{j_2} + W_{n+1}\boldsymbol{j_3}) \\
&\quad + t(W_{n-3} + W_{n-2}\boldsymbol{j_1} + W_{n-1}\boldsymbol{j_2} + W_n\boldsymbol{j_3}) \\
&= rHW_{n-1} + sHW_{n-2} + tHW_{n-3}.
\end{aligned}$$

□

The next lemma shows the relationship between hyperbolic generalized tribonacci quaternions and their conjugates:

Lemma 3.3. *Assume $n \geq 0$, the following equalities are valid:*

$$HW_n + \overline{HW_n} = 2W_n \tag{3.3}$$

$$HW_n - \overline{HW_n} = 2(W_{n+1}j_1 + W_{n+2}j_2 + W_{n+3}j_3) \tag{3.4}$$

$$HW_n\overline{HW_n} = W_n^2 - W_{n+1}^2 - W_{n+2}^2 - W_{n+3}^2. \tag{3.5}$$

Proof. It is evident that equation (3.3)-(3.4) hold. we will demonstrate the validity of equation (3.5). By (3.1), we have

$$\begin{aligned}
&HW_n\overline{HW_n} \\
&= (W_n + W_{n+1}\boldsymbol{j_1} + W_{n+2}\boldsymbol{j_2} + W_{n+3}\boldsymbol{j_3})(W_n - W_{n+1}\boldsymbol{j_1} - W_{n+2}\boldsymbol{j_2} - W_{n+3}\boldsymbol{j_3}) \\
&= W_nW_n - W_nW_{n+1}\boldsymbol{j_1} - W_nW_{n+2}\boldsymbol{j_2} - W_nW_{n+3}\boldsymbol{j_3} + W_{n+1}\boldsymbol{j_1}W_n \\
&\quad - W_{n+1}\boldsymbol{j_1}W_{n+1}\boldsymbol{j_1} - W_{n+1}\boldsymbol{j_1}W_{n+2}\boldsymbol{j_2} - W_{n+1}\boldsymbol{j_1}W_{n+3}\boldsymbol{j_3} + W_{n+2}\boldsymbol{j_2}W_n \\
&\quad - W_{n+2}\boldsymbol{j_2}W_{n+1}\boldsymbol{j_1} - W_{n+2}\boldsymbol{j_2}W_{n+2}\boldsymbol{j_2} - W_{n+2}\boldsymbol{j_2}W_{n+3}\boldsymbol{j_3} + W_{n+3}\boldsymbol{j_3}W_n \\
&\quad - W_{n+3}\boldsymbol{j_3}W_{n+1}\boldsymbol{j_1} - W_{n+3}\boldsymbol{j_3}W_{n+2}\boldsymbol{j_2} - W_{n+3}\boldsymbol{j_3}W_{n+3}\boldsymbol{j_3} \\
&= W_n^2 - W_{n+1}^2\boldsymbol{j_1^2} - W_{n+2}^2\boldsymbol{j_2^2} - W_{n+3}^2\boldsymbol{j_3^2} \\
&= W_n^2 - W_{n+1}^2 - W_{n+2}^2 - W_{n+3}^2.
\end{aligned}$$

□

Theorem 3.4. *(Binet formula for hyperbolic generalized tribonacci quaternions) Let α, β, γ be the roots of $x^3 - rx^2 - sx - t = 0$. Assume $n \geq 0$, then*

$$HW_n = \frac{d_1\alpha^n\hat{\alpha}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n\hat{\beta}}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n\hat{\gamma}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{3.6}$$

where $\hat{\alpha} = 1 + \alpha\boldsymbol{j_1} + \alpha^2\boldsymbol{j_2} + \alpha^3\boldsymbol{j_3}$, $\hat{\beta} = 1 + \beta\boldsymbol{j_1} + \beta^2\boldsymbol{j_2} + \beta^3\boldsymbol{j_3}$ *and* $\hat{\gamma} = 1 + \gamma\boldsymbol{j_1} + \gamma^2\boldsymbol{j_2} + \gamma^3\boldsymbol{j_3}$.

Proof. By using (2.2) and (3.1), we have

$$\begin{aligned}
HW_n &= W_n + W_{n+1}\boldsymbol{j_1} + W_{n+2}\boldsymbol{j_2} + W_{n+3}\boldsymbol{j_3} \\
&= \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\
&\quad + \left(\frac{d_1\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right)\boldsymbol{j_1} \\
&\quad + \left(\frac{d_1\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)}\right)\boldsymbol{j_2} \\
&\quad + \left(\frac{d_1\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)}\right)\boldsymbol{j_3} \\
&= \left(\frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_1\alpha^{n+1}\boldsymbol{j_1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_1\alpha^{n+2}\boldsymbol{j_2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_1\alpha^{n+3}\boldsymbol{j_3}}{(\alpha-\beta)(\alpha-\gamma)}\right) \\
&\quad + \left(\frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_2\beta^{n+1}\boldsymbol{j_1}}{(\beta-\alpha)(\beta-\gamma))} + \frac{d_2\beta^{n+2}\boldsymbol{j_2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_2\beta^{n+3}\boldsymbol{j_3}}{(\beta-\alpha)(\beta-\gamma)}\right) \\
&\quad + \left(\frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} + \frac{d_3\gamma^{n+1}\boldsymbol{j_1}}{(\gamma-\alpha)(\gamma-\beta)} + \frac{d_3\gamma^{n+2}\boldsymbol{j_2}}{(\gamma-\alpha)(\gamma-\beta)} + \frac{d_3\gamma^{n+3}\boldsymbol{j_3}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\
&= \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}\left(1 + \alpha\boldsymbol{j_1} + \alpha^2\boldsymbol{j_2} + \alpha^3\boldsymbol{j_3}\right) + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}\left(1 + \beta\boldsymbol{j_1} + \beta^2\boldsymbol{j_2} + \beta^3\boldsymbol{j_3}\right) \\
&\quad + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\left(1 + \gamma\boldsymbol{j_1} + \gamma^2\boldsymbol{j_2} + \gamma^3\boldsymbol{j_3}\right) \\
&= \frac{d_1\alpha^n\hat{\alpha}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n\hat{\beta}}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n\hat{\gamma}}{(\gamma-\alpha)(\gamma-\beta)}.
\end{aligned}$$

□

Theorem 3.5. *The summation for the first n+1 terms of hyperbolic generalized tribonacci quaternions is*

$$\sum_{i=0}^{n} HW_i = \frac{1}{\delta}\left(HW_{n+2} + (1-r)HW_{n+1} + tHW_n + \omega\right), \tag{3.7}$$

$$\begin{aligned}
\textit{where}\quad \delta &= \delta(r,s,t) = r + s + t - 1, \\
\lambda &= \lambda(r,s,t) = (r+s-1)a + (r-1)b - c, \\
\omega &= \omega(r,s,t) = \lambda + (\lambda - \delta a)\boldsymbol{j_1} + (\lambda - \delta(a+b))\boldsymbol{j_2} + (\lambda - \delta(a+b+c))\boldsymbol{j_3}.
\end{aligned}$$

Proof. By using (3.1), we have

$$\begin{aligned}
&\sum_{i=0}^{n} HW_i = HW_0 + HW_1 + HW_2 + ... + HW_n \\
&= (W_0 + W_1\boldsymbol{j_1} + W_2\boldsymbol{j_2} + W_3\boldsymbol{j_3}) + (W_1 + W_2\boldsymbol{j_1} + W_3\boldsymbol{j_2} + W_4\boldsymbol{j_3}) \\
&\quad + ... + (W_n + W_{n+1}\boldsymbol{j_1} + W_{n+2}\boldsymbol{j_2} + W_{n+3}\boldsymbol{j_3}) \\
&= (W_0 + W_1 + W_2 + ... + W_n) + (W_1 + W_2 + W_3 + ... + W_{n+1})\boldsymbol{j_1} \\
&\quad + (W_2 + W_3 + W_{k,4} + ... + W_{n+2})\boldsymbol{j_2} + (W_3 + W_4 + W_5 + ... + W_{n+3})\boldsymbol{j_3} \\
&= \sum_{i=0}^{n} W_n + (\sum_{i=0}^{n+1} W_n - W_0)\boldsymbol{j_1} + (\sum_{i=0}^{n+2} W_n - \sum_{i=0}^{1} W_n)\boldsymbol{j_2} + (\sum_{l=0}^{n+3} W_n - \sum_{i=0}^{2} W_n)\boldsymbol{j_3}.
\end{aligned}$$

From (2.3), we can write

$$\begin{aligned}
\delta\sum_{i=0}^{n} HW_i &= W_{n+2} + (1-r)W_{n+1} + tW_n + \lambda \\
&\quad + (W_{n+3} + (1-r)W_{n+2} + tW_{n+1} + \lambda - \delta a)\boldsymbol{j_1} \\
&\quad + (W_{n+4} + (1-r)W_{n+3} + tW_{n+2} + \lambda - \delta(a+b))\boldsymbol{j_2} \\
&\quad + (W_{n+5} + (1-r)W_{n+4} + tW_{n+3} + \lambda - \delta(a+b+c))\boldsymbol{j_3} \\
&= (W_{n+2} + W_{n+3}\boldsymbol{j_1} + W_{n+4}\boldsymbol{j_2} + W_{n+5}\boldsymbol{j_3}) \\
&\quad + ((1-r)W_{n+1} + (1-r)W_{n+2}\boldsymbol{j_1} + (1-r)W_{n+3}\boldsymbol{j_2} + (1-r)W_{n+4}\boldsymbol{j_3}) \\
&\quad + (tW_n + tW_{n+1}\boldsymbol{j_1} + tW_{n+2}\boldsymbol{j_2} + tW_{n+3}\boldsymbol{j_3}) \\
&\quad + \lambda + (\lambda - \delta a)\boldsymbol{j_1} + (\lambda - \delta(a+b))\boldsymbol{j_2} + (\lambda - \delta(a+b+c))\boldsymbol{j_3} \\
&= HW_{n+2} + (1-r)HW_{n+1} + tHW_n + \omega,
\end{aligned}$$

Finally,

$$\sum_{i=0}^{n} HW_i = \frac{1}{\delta}\left(HW_{n+2} + (1-r)HW_{n+1} + tHW_n + \omega\right).$$

□

Theorem 3.6. *The generating function for hyperbolic generalized tribonacci quaternions is*

$$\sum_{n=0}^{\infty} HW_n x^n = \frac{HW_0 + x(HW_1 - rHW_0) + x^2(HW_2 - rHW_1 - sHW_0)}{1 - rx - sx^2 - tx^3}. \tag{3.8}$$

Proof. Suppose that the generating function of the hyperbolic generalized tribonacci quaternions HW_n has the form $f(x) = \sum_{n=0}^{\infty} HW_n x^n$. Then

$$f(x) = HW_0 + HW_1x + HW_2x^2 + HW_3x^3 + ... + HW_nx^n + ...$$

Multiplying $f(x)$ on both side by rs, sx^2 and then tx^3, we have

$$\begin{aligned}
rxf(x) &= rHW_0x + rHW_1x^2 + rHW_2x^3 + ... + rHW_{n-1}x^n + rHW_nx^{n+1} + ... \\
sx^2f(x) &= sHW_0x^2 + sHW_1x^3 + sHW_2x^4 + ... + sHW_{n-1}x^{n+1} + sHW_nx^{n+2} + ... \\
tx^3f(x) &= tHW_0x^3 + tHW_1x^4 + tHW_2x^5 + tHW_3x^3 + ... + tHW_{n-1}x^{n+2} + tHW_nx^{n+3} ...
\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}(1-rx-sx^2-tx^3)f(x) &= (HW_0+HW_1x+HW_2x^2+HW_3x^3+...+HW_nx^n...)\\ &- (rHW_0x+rHW_1x^2+rHW_2x^3+...+rHW_{n-1}x^n+rHW_nx^{n+1}+...)\\ &- (sHW_0x^2+sHW_1x^3+sHW_2x^4+...+sHW_{n-1}x^{n+1}+sHW_nx^{n+2}+...)\\ &- (tHW_0x^3+tHW_1x^4+tHW_2x^5+tHW_3x^3+...+tHW_{n-1}x^{n+2}+tHW_nx^{n+3}...)\\ &= HW_0-x(-HW_1+rHW_0)-x^2(-HW_2+rHW_1+sHW_0)\\ &- \sum_{i=3}^{\infty}(-HW_i+(rHW_{i-1}+sHW_{i-2}+tHW_{i-3}))x^i\\ &= HW_0-x(-HW_1+rHW_0)-x^2(-HW_2+rHW_1+sHW_0)-\sum_{i=3}^{\infty}(-HW_i+HW_i)x^i\\ &= HW_0+x(HW_1-rHW_0)+x^2(HW_2-rHW_1-sHW_0).\end{aligned}$$

Therefore,

$$f(x)=\frac{HW_0+x(HW_1-rHW_0)+x^2(HW_2-rHW_1-sHW_0)}{1-rx-sx^2-tx^3}.$$

□

References

[1] Aydin, F. T. (2018). Hyperbolic k-Fibonacci quaternions. Journal of Computer Science and Computational Mathematics, 9(4).

[2] Cerda-Morales, G. (2017). On a generalization for tribonacci quaternions. Mediterranean journal of mathematics, 14, 1-12.

[3] Choi, E. M. (2013). Modular tribonacci numbers by matrix method. Pure and applied mathematics, 20(3), 207-221.

[4] Demir, S., Tanışlı, M., & Candemir, N. (2010). Hyperbolic quaternion formulation of electromagnetism. Advances in Applied Clifford Algebras, 20, 547-563.

[5] Elia, M. (2001). Derived sequences, tribonacci recurrence and cubic forms. The Fibonacci Quarterly, 37, 107-115.

[6] Feinberg, M. (1963). Fibonacci-tribonacci. The Fibonacci QSuarterly, 1(3), 71–74.

[7] Flaut, C. & Shpakivskyi, V. (2012). On generalized Fibonacci quaternions and Fibonacci-Narayana quaternions. arXiv preprint arXiv:1209.0584

[8] Hamilton, W. R. (1866). Elements of quaternions. London: Longmans, Green, & Company.

[9] Horadam, A. F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. The American Mathematical Monthly, 70(3), 289-291.

[10] Kösal,, İ. A. (2018). A note on hyperbolic quaternions. Universal journal of mathematics and Applications, 1(3), 155-159.

[11] Kuloğlu, B., Özkan, E. (2023). A new approach to Narayana hybrid sequences. 9th International Scientific Researches Conference, Conference Paper February.

[12] Macfarlane, A. (1902). Hyperbolic quaternions. Proceedings of the Royal Society of Edinburgh, 23, 169-180.

[13] Scott, A., DEL ANEY, T.O.M., & Hoggatt Jr, V.E. (1977). THE TR1B0NACCI SEQUENCE.

[14] Soykan, Y. (2020). On generalized Narayana numbers. Int. J. Adv. Appl. Math. and Mech, 7(3), 43-56.

[15] Takahashi, K. (2021). Comparison of high-dimensional neural networks using hypercomplex numbers in a robot manipulator control. Artificial Life and Robotics, 26(3), 367-377.

[16] Yağmur, T. (2020). A note on hyperbolic Fibonacci quaternions. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69(1), 880-890.

[17] Yalavigi, C.C. (1972). Properties of tribonacci Numbers. The Fibonacci Quarterly, 10(3), 231–246.

[18] Yilmaz, N., & Taskara, N. (2014). tribonacci and tribonacci-Lucas numbers via the determinants of special matrices. Applied Mathematical Sciences, 8(39), 1947-1955.

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